Review: Sets and Relations

The foundational mathematical objects we'll be working with in this course are sets.

Definition (Set)

A **set** is a collection of things.

If we are writing out a description of a set, we denote the set with curly braces. For convenience we sometimes give sets shorthand names like S. A set is very general and can contain *any* type of object ... even other sets!

Example. $S = \{4, \Delta, \Pi, \{2, \square\}\}$ is a set, and it contains the set $\{2, \square\}$.

The set in this example contains 4 objects, which we call **elements**.

We can also write $S = \{x : P(x) \text{ is true}\}$, where P is some statement about x. *The colon (:) should be read as "such that." S is the set of elements x such that the statement $P(x)$ is true. We can also use a vertical bar (|) to denote the same idea.*

Example. $S = \{x : x < 2\}$.

Below is some helpful shorthand notation for sets that we'll encounter a lot in this class.

Notation	Description
$x \in S$	x is in S
$x \notin S$	x is not in S
$\emptyset$	The empty set (the set with no elements)
$A \subset B$	A is a subset of B: if $x \in A$, then $x \in B$. Be careful ... A and B could be the same set!
$A \subseteq B$	Equivalent to $A \subset B$, but clearly denoting the possibility of equality of the sets.
$A \not\subset B$	A is not a subset of B
$A \cup B$	Union: The set $\{x : x \in A \text{ or } x \in B\}$
$A \cap B$	Intersection: The set $\{x : x \in A \text{ and } x \in B\}$
A^C $A \setminus B$	Complement: The set $\{x : x \notin A\}$ Set minus: The set $\{x : x \in A \text{ and } x \notin B\}$
$A \times B$	Cartesian product: The set $\{(a, b) : a \in A \text{ and } b \in B\}$. Here (a, b) is an ordered pair, so order matters.

In this course, we'll often find ourselves in the situation where we would like to show that two sets A and B are the same set. Set inclusion gives us a technique to do this. If $A \subset B$ and $B \subset A$, then it must be true that every element in A is an element of B and vice versa, that is, $A = B$. Otherwise, $A \neq B$.

If $A \subset B$ but $B \not\subset A$, we say that A is a **proper subset** of B.

Definition (Relation)

A (binary) **relation** R is a subset of another object, say $A \times B$, where if $(a, b) \in R$ we write aRb.

Examples:

- If P is the set of people, then A "is a student of" is a relation on $P \times P$. In the notation above:
$$\text{(you)}\, A\, \text{(Prof. Heather)}.$$
- If $\mathbb{Z}$ is the set of integers, then $<$ is a relation on $\mathbb{Z} \times \mathbb{Z}$. In the notation above:
$$7 < 10.$$

One important and useful type of relation is an **equivalence relation**, which satisfies some special properties.

Definition (Equivalence relation)

An **equivalence relation** R on a set S is a relation on $S \times S$ such that

1. aRa (R is *reflexive*)

2. aRb implies bRa (R is *symmetric*)

3. If aRb and bRc, then aRc (R is *transitive*)

Equivalence relations are often denoted with $\sim$ or $\simeq$ (we usually reserve $=$ for the identity).