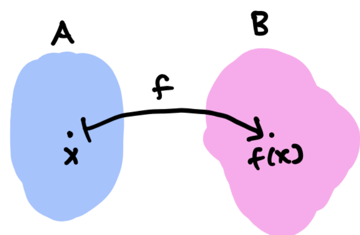
Review: Functions

A function takes elements of one set (called the **domain**) and maps them to another set (called the **codomain**) so that we get a unique output for each input. We use the following notation for a function f with domain A and codomain B :

$$\begin{aligned} f : A &\to B \\ x &\mapsto f(x). \end{aligned}$$

The top line gives the function's name along with the domain and codomain. Here, we use the regular arrow $\to$ (LaTeX `\to`). The bottom line indicates what happens to individual elements: an element x in A is mapped to an element $f(x)$ in B, with the 'maps to' arrow $\mapsto$ (LaTeX `\mapsto`).



Definition (Image and Inverse Image)

Consider a function $f : A \to B$. Suppose $C \subseteq A$ and $D \subseteq B$.

- We define $f(C) = \{f(x) : x \in C\}$ to be the **image** of C under this mapping.
- We define $f^{-1}(D) = \{x : f(x) \in D\}$ to be the **inverse image** of D under this mapping. Note that the inverse image may not be a unique x for each $f(x)$.

That is, the image a set of outputs $f(C)$ for a particular input set C. The inverse image is the set of inputs $f^{-1}D$ that yield a particular output set D. The image for the domain of a function is called the **range**.

Activity:

Explain in your own words the difference between codomain, range, and image. When are these quantities the same? When are they different? Give examples and/or draw pictures to illustrate this.

Next, we give some important terminology to describe a function $f : A \to B$.

Definition (Surjection)

When $f(A) = B$, we say that f is a **surjection** or that f is **surjective**. Alternatively, we say that f is **onto**.

In other words, a function is surjective is the range and the codomain are the same: it's a function whose mapping 'hits' every element in the codomain as an output.

Definition (Injection)

If $f(x) = f(y)$ implies that $x = y$, we say f is an **injection** or that f is **injective**. Alternatively, we say that f is **one-to-one**.

Equivalently, f is injective if the inverse image of $f(x)$ is a single point. In an injective function, each output came from a unique input.

Definition (Bijection)

If f is both one to one and onto, we say that f is a **bijection** or that f is **bijective**. A bijection means that the inverse of a function is also a function, and in this case we say that A and B are in **one-to-one correspondence**.

Activity:

Give examples and/or draw pictures of functions that satisfy the following:

- surjective, but not injective
- injective, but not surjective
- neither injective nor surjective
- bijective