Proof 1.

For this problem, you may use Rudin 1.33 (the standard Triangle Inequality).

Rudin 1.12

If $z_1, \ldots, z_n$ are complex, prove that

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$

This is a proof by induction on the variable n.

Starting with the base case n = 1. When n = 1, the inequality is $|z_1| \leq |z_1|$, which is true.

My inductive hypothesis is this true for n = k, such that $|z_1+z_2+...+z_k| \leq |z_1|+|z_2|+...+|z_k|$

To prove n = k + 1 is also true, I will use Rudin 1.33 (the standard Triangle Inequality). First I will take the left side of my inductive hypothesis, and extend the inside of the absolute value signs to n = k + 1 by adding z_{k+1} to the inside and adding $|z_{k+1}|$ to the outside to get $|z_1 + z_2 + ... + z_k + z_{k+1}|$ and $|z_1 + z_2 + ... + z_k| + |z_{k+1}|$. $|z_1 + z_2 + ... + z_k + z_{k+1}| \leq |z_1+z_2+...+z_k|+|z_{k+1}|$ because Rudin 1.33 states $|z+w| \leq |z|+|w|$, where $z_1+z_2+...+z_k = z$ and $z_{k+1} = w$. Now I will take my inductive hypothesis and just add the term $|z_{k+1}|$ to both sides to get $|z_1+z_2+...+z_k|+|z_{k+1}| \leq |z_1|+|z_2|+...+|z_k|+|z_{k+1}|$. Since $|z_1+z_2+...+z_k+z_{k+1}| \leq |z_1 + z_2 + ... + z_k| + |z_{k+1}|$ and $|z_1 + z_2 + ... + z_k| + |z_{k+1}| \leq |z_1| + |z_2| + ... + |z_k| + |z_{k+1}|$, by the transitive property, $|z_1 + z_2 + ... + z_k + z_{k+1}| \leq |z_1| + |z_2| + ... + |z_k| + |z_{k+1}|$ showing my inductive hypothesis is true.

We can now conclude that if $z_1, ..., z_n$ are complex, then $|z_1+z_2+...+z_n| \leq |z_1|+|z_2|+...+|z_n|$.

Proof 2.

Rudin 1.12

If $z_1, \ldots, z_n$ are complex, prove that

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$

Recall that the absolute value of a given complex number z_k, $k \in \mathbb{N}$ can be found as the non-negative square root of the product of itself and its conjugate, written

$$|z_k| = \sqrt{z_k \overline{z_k}}.$$

We can square the lefthand side and distribute

$$\begin{aligned} |z_1 + z_2 + \ldots + z_n|^2 &= (z_1 + z_2 + \ldots + z_n)(\overline{z_1} + \overline{z_2} + \ldots + \overline{z_n}) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} z_i \overline{z_j}. \end{aligned}$$

For a given pair of complex numbers z_i and z_j, $i,j \in \mathbb{N}$, recall that $z_i \overline{z_j}$ and $\overline{z_i} z_j$ are conjugates. Thus, $z_i \overline{z_j} + \overline{z_i} z_j = 2\text{Re}(z_i \overline{z_j})$. We can separate out the terms for which $i \neq j$, combine pairs by symmetry, and return to using absolute value notation

$$\begin{aligned} |z_1 + z_2 + \ldots + z_n|^2 &= \sum_{i=1}^{n} z_i \overline{z_i} + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} z_i \overline{z_j} \\ &= \sum_{i=1}^{n} z_i \overline{z_i} + \sum_{i=1}^{j} \sum_{j=1}^{n} 2\text{Re}(z_i \overline{z_j}) \\ &= \sum_{i=1}^{n} |z_i|^2 + \sum_{i=1}^{j} \sum_{j=1}^{n} 2\text{Re}(z_i \overline{z_j}). \end{aligned}$$

The real part of a complex number is always less than or equal to the absolute value, which also includes the magnitude of the imaginary part. Hence, we can state

$$\begin{aligned} |z_1 + z_2 + \ldots + z_n|^2 &= \sum_{i=1}^{n} |z_i|^2 + \sum_{i=1}^{j} \sum_{j=1}^{n} 2\text{Re}(z_i \overline{z_j}) \\ &\leq \sum_{i=1}^{n} |z_i|^2 + \sum_{i=1}^{j} \sum_{j=1}^{n} 2|z_i \overline{z_j}| \end{aligned}$$

For a pair of complex number z_i and z_j, it is true that $|z_i z_j| = |z_i||z_j|$ and that $|\overline{z}| = |z|$. We can use these identities to rewrite the second sum, which allows us to factor the entire expression

$$\begin{aligned} |z_1 + z_2 + \ldots + z_n|^2 &\leq \sum_{i=1}^{n} |z_i|^2 + \sum_{i=1}^{j} \sum_{j=1}^{n} 2|z_i||z_j| \\ &\leq (|z_1| + |z_2| + \ldots + |z_n|)^2 \end{aligned}$$

Taking the square root, we conclude that

$$|z_1 + z_2 + \ldots + z_n| \leq |z_1| + |z_2| + \ldots + |z_n|.$$

Proof 3.

Rudin 1.12

If $z_1, \ldots, z_n$ are complex, prove that

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$

Proof. We proceed by induction on n, the indices of the complex numbers $z_1, \ldots, z_n$.

We begin with the base case $n = 1$. Since $z_1 = z_1$, it follows that $|z_1| \leq |z_1|$. Therefore, the statement is true for $n = 1$.

Proceeding with the inductive step, assume the statement is true for some $n \in \mathbb{N}$, that is $|z_1 + \cdots + z_n| \leq |z_1| + \cdots + |z_n|$. We want to show that the statement also holds for $n + 1$. Consider

$$|z_1 + \cdots + z_n + z_{n+1}|$$

Letting $S_n = z_1 + \cdots + z_n$, the standard triangle inequality (Rudin 1.33 (e)) states that the following inequality is true:

$$|S_n + z_{n+1}| \leq |S_n| + |z_{n+1}|$$

We can apply the inductive hypothesis to the sum S_n as there are n complex numbers in the sum. Thus, by the inductive hypothesis, $|S_n| \leq |z_1| + |z_2| + \cdots + |z_n|$. Therefore, the following inequality is true about $|S_n| + |z_{n+1}|$:

$$|S_n| + |z_{n+1}| \leq |z_1| + \cdots + |z_n| + |z_{n+1}|$$

Since $S_n = z_1 + \cdots + z_n$, we have shown that $|z_1 + \cdots + z_n + z_{n+1}| \leq |z_1| + \cdots + |z_n| + |z_{n+1}|$, as desired. Thus, we have proven by induction that $|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$. $\square$

Proof 4.

Rudin 1.12

If $z_1, \ldots, z_n$ are complex, prove that

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n| .$$

The proof is by induction on n. For $n = 1$, we have $|z_1| \leq |z_1|$ which is trivially true. In the inductive case, let $n = k + 1$ where the statement holds for k and let $z = z_1 + \cdots + z_k$. If we add $|z_{k+1}|$ to both sides of the inductive hypothesis, we see that

$$|z| + |z_{k+1}| \leq |z_1| + \cdots + |z_k| + |z_{k+1}|.$$

From Theorem 1.33(e) in Rudin, we know that $|z + z_{k+1}| \leq |z| + |z_{k+1}|$. Thus, putting the two inequalities together, we have

$$|z + z_{k+1}| \leq |z| + |z_{k+1}| \leq |z_1| + \cdots + |z_k| + |z_{k+1}|,$$

so the statement holds by transitivity. Because the base case and induceive case hold, the statement is true for all n.