# Proof by induction

Proof by induction is a popular and useful proof technique. Here at Harvey Mudd College, we encounter it in our very first week in Math 19. In this lecture, we'll learn a little bit more about where this technique comes from and why it works, and we'll encounter some style tips and pitfalls to avoid.

## The natural numbers $\mathbb{N}$

What are the natural numbers? We can intuitively think of this set as the 'counting numbers' $\mathbb{N} = \{1, 2, 3, \dots\}$. A set of axioms for the natural numbers comes from Giuseppe Peano (1889).

**Definition (The Peano Axioms)**

Let $\mathbb{N}$ be a set containing the element 1. Define a successor function $S : \mathbb{N} \to \mathbb{N}$, with the following requirements:

1. $S(x) \neq 1$ for all $x \in \mathbb{N}$.
2. If $S(x) = S(y)$ then $x = y$ for all $x, y \in \mathbb{N}$.
3. Let $A$ be any subset of $\mathbb{N}$ which contains 1 and is closed under $S$, so $S(x) \in A$ for all $x \in A$.

Then $A = \mathbb{N}$.

The definition boils down to two ideas: there is a first element (1), and every element in the set has a successor in the set that is not 1. So, we can see that proof by induction is based on axiomatic properties of $\mathbb{N}$.

**Remark.** You can start $\mathbb{N}$ with zero instead of 1. I'm not going to take a stance on this debate either way.

The principle of induction implies the **well-ordering principle** of $\mathbb{N}$ :

**Definition (The well-ordering principle)**

Every nonempty subset of $\mathbb{N}$ has a smallest element.

## The basics of proof by induction

Proof by induction is a way to prove infinitely many statements at once, given that these statements can be indexed by $n \in \mathbb{N}$ : in other words, there's a first statement, a second statement, a third statement, and so on.

We can formalize this a little more: if we let $P(n)$ be a set of statements that are indexed by $n \in \mathbb{N}$, then the goal is to show that $P(n)$ is true for all $n$.

**How proof by induction works.** If $S = \{n : P(n) \text{ is true}\}$, where $n \in \mathbb{N}$, then is a subset of $\mathbb{N}$. If our goal is to show that $P(n)$ is true for all $n$, this is equivalent to showing that $S = \mathbb{N}$. To do this, we must show:

1. $P(1)$ is true. We call this the **base case.** (*There is a first element*)
2. If $P(k)$ is true then $P(k+1)$ is true. We call this the **inductive step**, and the assumption that $P(k)$ is true is called the **inductive hypothesis.** (*Every element has a successor*).

Then, because $S$ satisfies the axioms, it must be that $S = \mathbb{N}$ and so we can conclude our statement $P(n)$ is true for all $n$.

## Strong induction

There is a twist on this proof technique that we call **strong induction.** To use strong induction, we must show:

1. **Base case:** $P(1)$ is true.
2. **Inductive step:** If $P(1), P(2), \ldots, P(k)$ is true, then $P(k+1)$ is true.

Notice that the only difference is that we have modified the inductive hypothesis to assume that all statements up to statement $k$ are true, instead of only assuming that the $k$th statement is true.

> Despite the name, strong induction does not give you a 'stronger' result. You are still showing that $S$ is the same as $\mathbb{N}$, so these techniques are in fact equivalent.

## Style tips for proof by induction

- At the beginning of your proof, inform your reader that it's a proof by induction on ____________, where you fill in the blank with your variable that will serve as an index.
- Separate out your hypotheses: tell your reader when you're doing the base case and when you're doing the inductive step. Make your inductive hypothesis clear.
- Remind your reader of the conclusion at the end.

In the following examples in this lecture, I put these style tips into practice.

## Common errors in using proof by induction

Proof by induction is a useful technique, but you must make sure to proceed carefully: there are some common traps that it's surprisingly easy to fall into. I will demonstrate a few of these by constructing some false proofs using proof by induction.

Let's practice: Critique this "proof."

> Theorem (An untrue 'theorem' about natural numbers)
>
> Every natural number $n$ is even.

**"Proof."** We proceed by induction on $n$. By way of strong induction, assume that all numbers less than or equal to $n$ are even. We want to show that $n+1$ is also even. Notice that

$$n + 1 = (n - 1) + 2.$$

By our inductive hypothesis, $n-1$ is even. So, $n+1$ is the sum of two even numbers, which is also even, as desired. □

### What's wrong with this proof?

We jumped straight to the inductive hypothesis and we forgot to prove the base case. This is a really important step: for this example, our base case is not satisfied for $n = 1$, and so the rest of our logic does not follow.

Critique this "proof."

**Theorem (A 'theorem' about horses)**

All horses are the same color.

**"Proof."** We proceed by induction on the number of horses $n$ in a set of horses.

We begin with the base case, $n = 1$. In a set with 1 horse, all horses are trivially the same color.

Now, let $S_n = \{h_1, h_2, \ldots, h_n\}$ be a set of $n$ horses. Assume $S$ satisfies the inductive hypothesis, that is, all horses in $S_n$ are the same color. If there is another horse of that color that's not in the set, then we could add it to create a new set $S_{n+1} = \{h_1, h_2, \ldots, h_n, h_{n+1}\}$ which contains $n + 1$ horses of the same color, as desired.

We can then conclude that a set of any size must contain only horses of the same color, and thus all horses are the same color. □

**What's wrong with this proof?**

A common mistake is to try to start with the smaller set and 'build up' to the bigger set in the inductive step. This often leads to constructing specific examples or cases, like we did here. You should start with the bigger set (e.g., a set of size n+1) and find the smaller set that satisfies the inductive hypothesis within it.

Let's try that last theorem again. Critique this "proof."

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We begin with the base case, $n = 1$. In a set with 1 horse, all horses are trivially the same color.

Let $S_n = \{h_1, h_2, \ldots, h_n\}$ be a set of $n$ horses. Assume $S_n$ satisfies the inductive hypothesis, that is, all horses in $S_n$ are the same color. Now, consider the set $S_{n+1} = \{h_1, h_2, \ldots, h_n, h_{n+1}\}$. Notice this set contains $S_n = \{h_1, h_2, \ldots, h_n\}$ and $S'_n = \{h_2, h_2, \ldots, h_{n+1}\}$. Both sets contain $n$ horses, so they are all the same color in each set by the inductive hypothesis. But, both sets contain some of the same horses (for example, $h_2$) and so all horses in $S_{n+1}$ must be the same color.

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**What's wrong with this proof?**

This one is more subtle! Even though we corrected our error from the previous attempt, we've accidentally assumed that we have at least three horses. If there are only two horses, the inductive step fails, because there are not overlapping sets.

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**"Proof."** We proceed by induction on the number of horses $n$ in a set of horses.

We begin with the base case, $n = 1$. In a set with 1 horse, all horses are trivially the same color.

Let $S_n = \{h_1, h_2, \ldots, h_n\}$ be a set of $n$ horses. Assume $S_n$ satisfies the inductive hypothesis, that is, all horses in $S_n$ are the same color. Now, consider the set $S_{n+1} = \{h_1, h_2, \ldots, h_n, h_{n+1}\}$. Notice this set contains $S_n = \{h_1, h_2, \ldots, h_n\}$ and $S_n' = \{h_2, h_2, \ldots, h_{n+1}\}$. Both sets contain $n$ horses, so they are all the same color in each set by the inductive hypothesis. But, both sets contain some of the same horses (for example, $h_2$) and so all horses in $S_{n+1}$ must be the same color.

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**Activity:**

Prove by induction that every $2^n \times 2^n$ chess board with one square removed can be tiled by L-shaped tiles made of three squares.